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On space–time manifolds carrying a skew symmetric killing vector field

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Abstract

We analyse the structural properties, from a geometrical point of view, of space–time manifolds carrying a skew symmetric Killing vector field.

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1. Introduction

Let (M, g) be a general space–time with usual differentiability conditions, and normed hyperbolic metric g (i.e. g is of index 1). We assume in this paper that M carries a non-null skew-symmetric Killing vector field C [1] (see also [2]) having as generative the timelike vector field e_4 of an orthonormal vector basis

$$\mathcal{O} = \text{vect}\{e_A | A = 1, \ldots, 4\}$$

on (M, g). Since we also assume that the covariant differential operator ∇ is symmetric, then ∇C is expressed by

$$\nabla C = C \wedge e_4, \tag{1}$$

where $\wedge$ is the wedge product of vectorfields.

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It has been shown that the existence of C is determined by a closed differential system and that e_4 is torse forming [3].

In the present paper we are mostly concerned with the case when $e_4^\flat$ is invariant under C. This implies that $\mathcal{D}_C = \{C, e_4\}$ defines an exterior concurrent 2-distribution [5] and any vector field $Z_C \in \mathcal{D}_C$ satisfies

$$\mathcal{R}(Z, Z_C) = 3g(Z, Z_C), \qquad Z \in \Xi M,$$

where $\mathcal{R}$ denotes the Ricci tensor field of ∇.

In addition, the following properties are proved:

(i) (M, g) is foliated by surfaces normal to $\mathcal{D}_C$;

(ii) the mixed curvature forms of M [15] define a vector valued 2-form ψ which is covariantly closed;

(iii) the square of the lengths $\|C\|^2$ is an isoparametric function.

It is well known that in special relativity, electromagnetism is described in the 4-vector formalism by the Maxwell tensor $(F^{\mu\nu})$, which incorporates both the electric and magnetic field [7]. Equivalently, in the language of forms, there thus exists a 2-form $\mathcal{F}$ on $\mathbb{R}^4$

$$\mathcal{F} = \sum_{\alpha=1}^{3} \mathcal{E}_a du^a \wedge du^4 + (\mathcal{B}_1 du^2 \wedge du^3 + \mathcal{B}_2 du^3 \wedge du^1 + \mathcal{B}_3 du^1 \wedge du^2),$$

where u^i $(i = 1, 2, 3, 4)$ are coordinates in Minkowski space [14].

Therefore, in general relativity on a space–time manifold M, electromagnetism is introduced by a 2-form on M

$$\mathcal{F} = (\mathcal{E}_a \omega^a) \wedge \omega^4 + \sum_{\text{cycl}} \mathcal{B}_a \omega^b \wedge \omega^c,$$

where $\{\omega^A | A = 1, \ldots, 4\}$ is a local field of orthonormal coframes over M; $\mathcal{F}$ is called the Faraday 2-form.

If one assumes that $\mathcal{B}$ and $\mathcal{E}$ coincide with the spacelike components of the skew-symmetric Killing vector field C, one finds that $\mathcal{F}$ is a conformal symplectic form having the timelike covector $e_4^\flat = \omega^4$ as covector of Lee.

Finally, regarding the Lie algebra induced by C and e_4, the following properties are pointed out:

(i) $C^\flat$ is self-invariant, i.e.

$$\mathcal{L}_C C^\flat = 0;$$

(ii) C is an affine Killing vector field;

(iii) the vector valued 2-form ψ is invariant by C, i.e.

$$\mathcal{L}_C \psi = 0;$$

(iv) $\mathcal{L}_C \mathcal{F}$ and $\mathcal{L}_{e_4} \mathcal{F}$ are cohomologically closed, i.e.

$$d^{-2\omega^4}(\mathcal{L}_C \mathcal{F}) = 0, \qquad d^{-2\omega^4}(\mathcal{L}_{e_4} \mathcal{F}) = 0.$$

2. Preliminaries

Let (M, g) be a Riemannian or pseudo-Riemannian C^∞-manifold and let ∇ be the covariant differential operator defined by the metric tensor g (we assume that ∇ is the Levi-Civita connection). Let ΓTM be the set of sections of the tangent bundle, and

$$TM \xrightarrow{\flat} T^*M \quad \text{and} \quad TM \xleftarrow{\sharp} T^*M$$

the classical isomorphism defined by g (i.e. $\flat$ is the index lowering operator, and $\sharp$ is the index raising operator).

We denote by

$$A^q(M, TM) = \Gamma\mathrm{Hom}(\Lambda^q TM, TM),$$

the set of vector valued q-forms ($q < \dim M$), and following [4], we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$$

(it should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$ unlike $d^2 = d \circ d = 0$). If $p \in M$ then the vector valued 1-form $dp \in A^1(M, TM)$ is the identity vector valued 1-form and is also called the soldering form of M [9]. Since ∇ is symmetric, one has that $d^\nabla(dp) = 0$.

The cohomology operator [8] is defined by

$$d^\omega = d + e(\omega) \tag{2}$$

and is acting on ΛM; $e(\omega)$ is the exterior product by the closed 1-form ω. One has $d^\omega \circ d^\omega = 0$, and a form $u \in \Lambda M$ such that $d^\omega u = 0$ is said to be d^ω-closed (or cohomologically closed).

A vector field $X \in \Xi M$ which satisfies

$$d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge dp \in A^2(M, TM), \quad \pi \in \Lambda^1 M \tag{3}$$

is defined to be an exterior concurrent vector field [1] (see also [2,6]). In (3) π is called the concurrence form and is defined by

$$\pi = \lambda X^\flat, \quad \lambda \in \Lambda^0 M. \tag{4}$$

In this case, if $\mathcal{R}$ is the Ricci tensor, one has

$$\mathcal{R}(X, Z) = \epsilon(n-1)\lambda g(X, Z), \tag{5}$$

($\epsilon = \pm 1$, $Z \in \Xi M$, $n = \dim M$).

A function $f : \mathbb{R}^n \to \mathbb{R}$ is isoparametric if $(df)^\sharp$ and $\mathrm{div}(df)^\sharp$ are functions of f [10] ($(df)^\sharp = \mathrm{grad}\, f$).

If $\mathcal{T}$ is any conformal vector field, i.e. the conformal version of the Killing equation, that is $\mathcal{L}_\mathcal{T} = \rho g$, $\rho = (2/n)\mathrm{div}\,\mathcal{T}$ then

$$\mathcal{L}_\mathcal{T} Z^\flat = \rho Z^\flat + [\mathcal{T}, Z]^\flat \tag{6}$$

is Oersteds lemma [11].

Let $\mathcal{O} = \{e_A | A = 1, \ldots, n\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \text{covect}\{\omega^A\}$ be its associated coframe. Then Cartans structure equations written in indexless manner are

$$\nabla e = \theta \otimes e, \tag{7}$$

$$d\omega = -\theta \wedge \omega, \tag{8}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{9}$$

In the above equations θ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature forms on M).

3. Skew-symmetric Killing vector fields

Let (M, g) be a space–time manifold with normal hyperbolic metric tensor g and let $\mathcal{O} = \{e_A | A = 1, \ldots, 4\}$ be a local field of orthonormal frames over M and $\mathcal{O}^* = \text{covect}\{\omega^A\}$ its associated coframe. We assume that the indices $a, b \in \{1, 2, 3\}$ correspond to the spacelike vectorfields e_A, whilst e_4 corresponds to the timelike vectorfield. Then by reference to [12] (see also [2]), the soldering form dp is expressed by:

$$dp = \omega^4 \otimes e_4 - \sum \omega^a \otimes e_a, \tag{10}$$

and in consequence of (10) one has the following structure equations [12]:

$$\nabla e_a = \theta_a^4 \otimes e_4 - \theta_a^b \otimes e_b, \qquad \nabla e_4 = -\theta_4^a \otimes e_a \tag{11}$$

$$d\omega^a = \omega^4 \wedge \theta_4^a - \omega^b \wedge \theta_b^a, \qquad d\omega^4 = -\omega^a \wedge \theta_a^4 \tag{12}$$

and

$$d\theta_b^a = \Theta_b^a - \theta_b^c \wedge \theta_c^a + \theta_b^4 \wedge \theta_4^a, \qquad d\theta_4^a = \Theta_4^a - \theta_4^b \wedge \theta_b^a. \tag{13}$$

Following [15], the curvature forms Θ_4^a are called the mixed curvature forms of the space–time manifold M.

Let now

$$C = C^4 e_4 - \sum C^a e_a, \quad C^A \in \Lambda^0 M, \tag{14}$$

be a vector field on M. In order to simplify, we set

$$C^b = \alpha = C^4 \omega^4 - \sum_a C^a \omega^4. \tag{15}$$

Assume that C is a skew-symmetric conformal vector field [13] (see also [13]), having the timelike vector field e_4 as generative, i.e.

$$\nabla C = 2\lambda dp + C \wedge e_4, \quad \lambda \in \Lambda^0 M. \tag{16}$$

As is known, one has $\mathcal{L}_C g = 2\lambda g$, $2\lambda = \operatorname{div} C/2$, and one may write (16) as

$$\nabla C = 2\lambda dp + \omega^4 \otimes C - \alpha \otimes e_4. \tag{17}$$

Making use of Eq. (11), one gets from (17)

$$\begin{aligned} dC^a &= C^b\theta_b^a - C^4\theta_4^a + C^a\omega^4 + 2\lambda\omega^a, \\ dC^4 &= C^b\theta_b^4 + (2\lambda + C^4)\omega^4 - \alpha \end{aligned} \tag{18}$$

and with the help of (12), one derives

$$d\alpha = 2\omega^4 \wedge \alpha. \tag{19}$$

But by definition, $d\omega^4$ is not conformal to α and so it follows at once from (19) that

$$d\omega^4 = 0. \tag{20}$$

Since α is an exterior recurrent form having $2\omega^4$ as recurrence form [17], we notice that (19) is the standard equation which is induced by skew-symmetric conformal or skew-symmetric Killing vector fields.

Without loss of generality, one may assume that the quadratic form $\langle dp, \nabla e_4\rangle$ is diagonal. Then one may set

$$\theta_a^4 = -f\omega^a = 0, \quad f \in \Lambda^0 M, \tag{21}$$

and this by (10) and the second equation of (11), implies

$$\nabla e_4 = f(dp - \omega^4 \otimes e_4), \tag{22}$$

which shows that the timelike vector field e_4 is torse forming [3] (see also [2,18]).

Now, by (20) one quickly has

$$\mathcal{L}_C\omega^4 = dC^4 = (f-1)\alpha + (2\lambda + (1-f)C^4)\omega^4, \tag{23}$$

and by Oersteds lemma (6), one finds

$$\mathcal{L}_C\omega^4 = (f-1)\alpha + (1-f)C^4\omega^4,$$

which implies that $2\lambda = 0$, i.e. C is a skew-symmetric Killing vectorfield.

Therefore we have the following proposition:

Proposition 3.1. *On a space–time manifold (M, g) a skew-symmetric conformal vector field C having as generative its timelike vectorfield e_4 of the $\mathcal{O}$-basis on M is a skew-symmetric Killing vector field.*

4. Exterior concurrent distribution

In consequence of Proposition 3.1, we will study the case when Eq. (16) reduces to

$$\nabla C = C \wedge e_4, \tag{24}$$

i.e. C is a skew-symmetric Killing vector field having the timelike vector field e_4 as generative. In addition we assume that ω^4 ($(\omega^4)^\sharp = e_4$) is invariant by C, i.e.

$$\mathcal{L}_C\omega^4 = 0 \iff C^4 = \text{const.} \tag{25}$$

In this case, it is seen by (23) that $f = 1$, (remember $\lambda = 0$) and the converse is also true. In consequence of (25), e_4 is a closed torse forming and, therefore, e_4 is an exterior concurrent vector field [1]. Moreover, since (24) may be expressed as

$$\nabla C = \omega^4 \otimes C - \alpha \otimes e_4, \tag{26}$$

then operating on ∇C by the operator d^∇, one derives

$$d^\nabla(\nabla C) = \nabla^2 C = \alpha \wedge dp, \tag{27}$$

which shows the fact that C is also an exterior concurrent vector field. One may prove that the converse is also true, that is:

The properties that the timelike covector ω^4 be invariant by C and that C be an exterior concurrent vector field are mutually equivalent.

It should be noticed that the existence of the skew-symmetric Killing vector field C is assumed by the closed differential system defined by (19) and (20). Since the operator ∇ acts inductively, one derives from (27) successively,

$$\nabla^3 C = 2\omega^4 \wedge \nabla^2 C, \qquad \nabla^4 C = 0.$$

Hence, one may say that $\nabla^3 C$ is covariant recurrent and that $\nabla^4 C$ is a zero element of $A^4(M, TM)$. Moreover, since e_4 and C are both exterior concurrent vector fields, and that this property is preserved by linearity, one may say that $\mathcal{D}_C = \{C, e_4\}$ defines an exterior concurrent 2-distribution. Since clearly $\mathcal{D}_C$ is an involutive distribution, it follows that M is foliated by surfaces $M_C^\perp$ normal to $\mathcal{D}_C$.

Next, since in general

$$\nabla^2 X(Z, Z') = R(Z, Z')X, \qquad \mathcal{R}(Z, X) = \sum \langle R(e_A, Z)X, e_A \rangle,$$

where R (resp. $\mathcal{R}$) denotes the curvature tensor field (resp. the Ricci tensor field of ∇), one finds for any vector field $X_C \in \mathcal{D}_C$

$$\mathcal{R}(Z, X_C) = 3g(Z, X_C), \quad Z \in \Xi M. \tag{28}$$

On the other hand, making use of the second structure equation (13), one derives by (21) and taking into account that $f = 1$

$$\Theta_a^4 = \omega^a \wedge \omega^4. \tag{29}$$

This shows that the mixed curvature forms Θ_a^4 are monomial and one easily finds

$$\sum C^a \Theta_a^4 = \omega^4 \wedge \alpha$$

and

$$\psi = \sum \Theta_4^a \otimes e_a = \omega^4 \wedge dp. \tag{30}$$

We agree to call ψ the mixed curvature vector valued 2-form of the space–time M under consideration. Operating on ψ by the operator d^∇ one finds on behalf of (20)

$$d^\nabla \psi = 0, \tag{31}$$

which shows that ψ is a closed vector valued 2-form. In another order of ideas, if we set

$$2l = \|C\|^2, \tag{32}$$

then since $dC^4 = 0$, one derives from (18)

$$dl = 2l\omega^4 - C^4\alpha, \tag{33}$$

which is coherent with (19). Hence one may write

$$(d2l)^\sharp = 4le_4 - C^4C \tag{34}$$

and we notice that $(d2l)^\sharp$ is also an exterior concurrent vector field. Next, since in general $\operatorname{div} Z = \operatorname{tr} \nabla Z$, $Z \in \Xi M$, then, taking into account the signature of g, one infers by (22)

$$\operatorname{div}(d2l)^\sharp = -4l - (C^4)^2. \tag{35}$$

Hence, since $\|(d2l)^\sharp\|^2 = 16l^2 - 6(C^4)^2 l$ and $C^4 = \text{const.}$, it follows that $\|(d2l)^\sharp\|^2$ and $\operatorname{div}(d2l)^\sharp$ are functions of $2l$. So, following the well-known definition [10], we conclude that $\|C\|^2$ is an isoparametric function. Moreover, setting in order to simplify

$$V = (d2l)^\sharp, \tag{36}$$

one finds by (22), (25) and (34)

$$\nabla V = 4l dp - 2C^4\omega^4 \otimes C + 2dl \otimes e_4, \tag{37}$$

and by a standard calculation it follows

$$[C, e_4] = 0, \tag{38}$$

$$[V, e_4] = -8le_4, \tag{39}$$

$$[V, C] = -\tfrac{1}{2} d\alpha(e_4, C)C. \tag{40}$$

This shows that C and e_4 commute and V defines an infinitesimal conformal transformation of e_4 and C. Finally, by reference to the Faraday 2-form $\mathcal{F}$ on a Minkowski space, we agree to denominate

$$\mathcal{F} = (\mathcal{E}_a\omega^a) \wedge \omega^4 + \sum_{\text{cycl}} \mathcal{B}_a\omega^b \wedge \omega^c, \tag{41}$$

(where $\sum_{\text{cycl}}$ is understood to be a cyclic permutation of the spacelike indices) the generalized Faraday 2-form (or the space–time Faraday form). The coefficients $\mathcal{E}_a$ and $\mathcal{B}_a$ represent the electric and magnetic fields, respectively, associated with $\mathcal{F}$. If we assume now that both vector fields $\mathcal{E}$ and $\mathcal{B}$ coincide with the spacelike components of C (i.e. $C_a = \mathcal{E}_a = \mathcal{B}_a$), then on behalf of (15) $\mathcal{F}$ is expressed by

$$\mathcal{F} = -\alpha \wedge \omega^4 + \phi, \tag{42}$$

where we have set

$$\phi = \sum_{\text{cycl}} C_a\omega^b \wedge \omega^c. \tag{43}$$

Next, with the help of (18), where $\lambda = 0$, and making use of the structure equations (12) and remembering that $f = 1$, one derives, after carrying out the calculations

$$d\phi = 3\omega^4 \wedge \phi. \tag{44}$$

Now, by (19) one quickly derives from (42) and (44)

$$d\mathcal{F} = 3\omega^4 \wedge \mathcal{F}, \tag{45}$$

which shows the fact that $\mathcal{F}$ is a conformal symplectic form, having up to 3, the timelike covector ω^4 as covector of Lee.

Further, taking the codifferential $\delta\mathcal{F}$ of $\mathcal{F}$, one finds, since $\delta = *d*$ ($*$: star operator with respect to the orientation of M), that

$$\delta\mathcal{F} = *d*\mathcal{F} = *d(-\phi + \alpha \wedge \omega^4) = - * d\mathcal{F},$$

and also

$$\delta\mathcal{F} = 3\alpha_s,$$

where α_s is the spacelike component of the Killing covector $C^\flat = \alpha$. Therefore, according to the well-known definition (see also [19]), $3\alpha_s$ may be considered as the source 1-form corresponding to $\mathcal{F}$.

Summarizing, we have the following theorem:

Theorem 4.1. *Let (M, g) be a general space–time manifold carrying a skew-symmetric Killing vector field C, having the timelike vector field e_4 of an $\mathcal{O}$-basis as generative. Such a vectorfield C is determined by a closed differential system and e_4 is torse forming.*

If $(e_4)^\flat$ is invariant by C, i.e. $\mathcal{L}_C(e_4)^\flat = 0$, then the distribution $\mathcal{D}_C = \{C, e_4\}$ is an exterior concurrent distribution and if $\mathcal{R}$ denotes the Ricci tensor field, then any vector field $Z_C \in \mathcal{D}_C$ satisfies

$$\mathcal{R}(Z, Z_C) = 3g(Z, Z_C), \quad Z \in \Xi M.$$

In addition, one has the following properties

(i) *M is foliated by surfaces normal to $\mathcal{D}_C$;*

(ii) *the square of the lengths $2l = \|C\|^2$ of C is an isoparametric function, and $(d2l)^\sharp$ defines an infinitesimal conformal transformation of C and e_4;*

(iii) *the mixed curvature forms Θ_a^4 are monomial and they define a vector valued 2-form ψ which is covariant closed, i.e. $d^\nabla\psi = 0$.*

Finally, define

$$\mathcal{F} = (\mathcal{E}_a\omega^a) \wedge \omega^4 + \sum_{\text{cycl}} \mathcal{B}_a\omega^b \wedge \omega^c$$

the generalized Faraday 2-form on the space–time (M, g) ($\sum_{\text{cycl}}$ denotes the cyclic permutation of the spacelike indices, and $\mathcal{E}$ (resp. $\mathcal{B}$) represents the electric (resp. the magnetic) vector field associated with $\mathcal{F}$). If $\mathcal{E}$ and $\mathcal{B}$ coincide with the spacelike components of C, *then $\mathcal{F}$ is a conformal symplectic form having up to 3 the timelike covector $\omega^4 = (e_4)^\flat$ as covector of Lee and $\delta F = 3(C_s^\flat)$ is the source 1-form corresponding to $\mathcal{F}$ ($C_s^\flat$: spacelike component of $C^\flat$).*

5. Properties of the Lie algebra

In this section, we shall point out some properties of the Lie algebra induced by the vector fields C, e_4 and V ($V = (d2l)^\sharp$). By (33) and (19) one derives after a short calculation

$$\mathcal{L}_C\alpha = 0 \quad (\alpha = C^\flat), \tag{46}$$

which shows that α is a self-invariant form.

Next, with the help of (46) and (38), one infers, taking the Lie derivative of (26)

$$\mathcal{L}_C\nabla C = 0. \tag{47}$$

Hence the above equation proves the fact that the skew-symmetric Killing vector field C under consideration is an affine Killing vector field (see also [4]). On the other hand, denote by

$$\sigma = \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4,$$

the volume element of M. Then taking the star operator $*\omega^4$ of ω^4 one has

$$*\omega^4 = -\omega^1 \wedge \omega^2 \wedge \omega^3 \tag{48}$$

and since ω^4 is closed, one readily finds

$$\mathcal{L}_{e_4} * \omega^4 = -3 * \omega^4.$$

Hence, one may say that ω^4 is star self-conformal. In the same order of ideas, it is easily seen that $\delta\omega^4 = 0$, (δ: codifferential operator) and so one may write

$$(d\delta + \delta d)\omega^4 \iff \Delta\omega^4 = 0.$$

Thus the space–time covector ω^4 is a harmonic form. On the other hand, by (30) and (31), one gets at once

$$\mathcal{L}_C\psi = 0,$$

which shows that the vector valued 2-form ψ is invariant under C. Further, on behalf of (43) and (42) it is easily seen that one has

$$i_C\phi = 0, \tag{49}$$

and

$$i_C\mathcal{F} = -2l\omega^4 \quad (2l = \|C\|^2). \tag{50}$$

In consequence of (49) and (50), it follows from (44) and (45) that

$$\mathcal{L}_C\mathcal{F} = -C^4(2\mathcal{F} + \alpha \wedge \omega^4), \tag{51}$$

and, by exterior differentiation of (51), one may write by reference to (2)

$$d^{-2\omega^4}(\mathcal{L}_C\mathcal{F}) = 0. \tag{52}$$

Hence, it turns out that the Lie derivative $\mathcal{L}_C\mathcal{F}$ is a $d^{-2\omega^4}$-closed form. Since $i_{e_4}\mathcal{F} = \alpha$, one similarly finds that $\mathcal{L}_{e_4}\mathcal{F}$ is also a $d^{-2\omega^4}$-closed form.

Finally, let $\mathbb{L} : u \to u \wedge \Omega$ be the operator of Weyl [16] ($u \in \Lambda^1 M, \Omega \in \Lambda^2 M$), and by iteration $u_q = \mathbb{L}^q u = u \wedge \Omega^q \in \Lambda^{2q+1} M$. Since in the case under consideration $\dim M = 4$, we consider the 3-form

$$\mathbb{L}^1\alpha = \alpha_1 = \alpha \wedge \mathcal{F}.$$

Then, on behalf of (46) and (51), one has

$$\mathcal{L}_C\alpha_1 = (\mathcal{L}_C\alpha) \wedge \mathcal{F} + \alpha \wedge \mathcal{L}_C\mathcal{F} = 2C^4\alpha_1.$$

Remembering that C^4=const., the above operation proves that C defines an infinitesimal homothety of the 3-form α_1. If we denote, $(\omega^4)_1 = \omega^4 \wedge \mathcal{F}$, one finds analogously that

$$\mathcal{L}_{e_4}(\omega^4)_1 = 2(\omega^4)_1,$$

which means that e_4 defines an infinitesimal homothety of the 3-form $(\omega^4)_1$.

Theorem 5.1. *Let (M, g) be a space–time manifold carrying a skew-symmetric Killing vector field C having the timelike vector field e_4 as generative. The following properties of the Lie algebra are induced by C and e_4:*

(i) *$C^\flat$ is a self-invariant form, i.e.*

$$\mathcal{L}_C C^\flat = 0;$$

(ii) *C is an affine Killing vector field, i.e.*

$$\mathcal{L}_C \nabla C = 0;$$

(iii) *the timelike covector field $\omega^4 = e_4^\flat$ is star self-conformal, i.e.*

$$\mathcal{L}_{e_4} * e_4^\flat = -3 * e_4^\flat;$$

(iv) *the vector valued 2-form $\psi \in A^2(M, TM)$ associated with the mixed curvature forms Θ_a^4 is invariant by C, i.e.*

$$\mathcal{L}_C \psi = 0;$$

(v) *the Lie derivatives $\mathcal{L}_C\mathcal{F}$ and $\mathcal{L}_{e_4}\mathcal{F}$ of the generalized Faraday 2-form are cohomologically closed, i.e.*

$$d^{-2\omega^4}(\mathcal{L}_C\mathcal{F}) = 0, \qquad d^{-2\omega^4}(\mathcal{L}_{e_4}\mathcal{F}) = 0,$$

and C and e_4 define infinitesimal homotheties of Weyls operator $\mathbb{L}$ associated with $\mathcal{F}$.

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